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KILLING VECTOR FIELDS AND HARMONIC FORMS

BY

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ABSTRACT. The paper is concerned with harmonic (p, q)-forms on compact Kähler manifolds which admit Killing vector fields with discrete zero sets. Let $h^{p,q}$ denote the dimension of the space of harmonic (p, q)-forms. The main theorem states that $h^{p,q} = 0$, $p \neq q$.

1. Throughout this paper M^n will denote a compact Kähler manifold of complex dimension n. We use elementary methods to show $h^{p,q} = 0$, if $p \neq q$ and M^n admits a Killing vector field with a discrete zero set. The integer $h^{p,q}$ is equal to the dimension of the harmonic (p, q)-forms.

I would like to thank G. Lusztig for suggesting the underpinning idea of this paper: relating the results of Frankel and Kosniowski to conclude the result. Special thanks are given to A. Howard for his many helpful conversations and observations.

Recently, J. B. Carrell and D. Lieberman [1] have generalized results of both A. Howard [3] and myself [8] by using the degeneracy criterion of Deligne to prove the following: If M^n admits a nontrivial holomorphic vector field Z with zeros, then $h^{p,q} = 0$ if $|p - q| > \dim_C \text{zero}(Z)$.

2. We start by fixing some notation. The real tangent bundle of M^n will be denoted by T. The bundle of complex vectors of type (1, 0) is denoted by $T^{(1,0)}$. If Y is any vector (real or complex), L_Y denotes the Lie derivative with respect to Y. By T^c we mean the complexification of T. If L is an endomorphism of T, L^c denotes the extension of L to T^c.

We now state a modified version of a theorem of T. Frankel.

2.1 THEOREM [2]. *Assume M^n admits a Killing field X with zero set* zero $(X) = \bigcup_\alpha F_\alpha$. *The F_α's are the connected components of the zero set. Then, there exists a Morse function f on M^n such that the nondegenerate critical manifolds of f are the F_α's. Furthermore, the index λ_α of f on F_α is twice the number of positive eigenvalues of*

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$$-iL^c_{X(p)}\colon T^{(1,0)}_p \longrightarrow T^{(1,0)}_p,$$

$p \in F_\alpha$ and $i = \sqrt{-1}$.

2.2 COROLLARY [2]. *If X has a discrete zero set, then $b_{2k+1}(M^n) = 0$, $k = 0, \cdots, n$. The integer $b_{2k+1}(M^n)$ is the $2k+1$ Betti number.*

3. Let Z be a holomorphic vector field with zero set $p_1, \cdots, p_m$. We say that Z has a simple isolated zero at p_α if $L_{Z(p_\alpha)}$ is an automorphism of $T^{(1,0)}_{p_\alpha}$. Thus, we can find a complex number c such that $\mathrm{Real}(\Theta/c) \neq 0$, Θ an eigenvalue of $L_{Z(p_\alpha)}$.

3.1 DEFINITION. Let Z be a holomorphic vector field with simple isolated zeros. The integer $s(\alpha, c)$ is the number of eigenvalues Θ of $L_{Z(p_\alpha)}$ such that $\mathrm{Real}(\Theta/c) > 0$.

A basic definition is now recalled.

3.2 DEFINITION. Let $\Omega^p(M^n)$ denote the sheaf of germs of holomorphic p-forms on M^n. Put $H^p(M^n, \Omega^p) = H^{p,q}(M^n)$. Define:

$$\chi^p(M^n) = \sum_q (-1)^q \dim_C H^{p,q}(M^n) = \sum_q (-1)^q h^{p,q}$$

and

$$\chi_y(M^n) = \sum_p \chi^p(M^n) y^p,$$

where y is an indeterminate.

We may now state Kosniowski's results

3.3 THEOREM [5]. *If Z is a holomorphic vector field with simple isolated zeros, then $\chi_y(M^n) = \Sigma_{p \in \mathrm{zero}(Z)} (-y)^{s(p,c)}$.*

3.4 COROLLARY [5]. *If Z is as above, then $(-1)^j \chi^j(M^n) \geqslant 0$, for all j. In fact, $(-1)^j \chi^j(M^n) =$ the number of points of $\mathrm{zero}(Z)$ such that $s(p, c) = j$.*

4. In order to relate the results of §§2 and 3, we must recall several elementary propositions. To this end, let J denote the complex structure of M^n.

4.1 PROPOSITION [2]. *If X is a Killing vector field on M^n, then $X - iJ(X)$ is a holomorphic vector field.*

4.2 PROPOSITION [4]. *Let W be a real vector field on M^n. Then, $L_W J = 0$ if and only if $L_W(JY) = J \cdot L_W(Y)$ for all real vector fields Y. Such a W is said to be an infinitesimal automorphism of J.*

4.3 PROPOSITION [4]. *The Lie algebra of infinitesimal automorphisms of J is isomorphic to the Lie algebra of holomorphic vector fields, the correspondence being $W \longrightarrow W - iJ(W)$.*

The following key lemma is proven by a straightforward calculation which employs 4.2.

4.4 LEMMA. *If* Z *is a holomorphic vector field, then on* $T_p^{(1,0)}$

$$-iL_{Z(p)} = -2iL^c_{X(p)},$$

where $Z = X - iJ(X)$, $p \in \text{zero}(Z)$.

4.5 PROPOSITION. *Suppose* X *is a Killing vector field with isolated zeros* $p_1, \cdots, p_m$. *Then,* $2s(p_\alpha, i) = \lambda_\alpha$.

PROOF. It is clear that $Z = X - iJ(X)$ is a holomorphic vector field (4.1) with simple isolated zeros at $p_1, \cdots, p_m$. But, by 4.4, $-iL_{Z(p_\alpha)} = -2iL^c_{X(p_\alpha)}$ on $T^{(1,0)}_{p_\alpha}$. Since X is Killing, $L^c_{X(p_\alpha)}$ is a skew-Hermitian endomorphism of $T^{(1,0)}_{p_\alpha}$. Thus, by the very definition of $s(p_\alpha, i)$ and 2.1 we conclude the proposition. Q.E.D.

5. We now have enough to prove the main theorem, but first we prove:

5.1 PROPOSITION [8]. *If* M^n *is a compact connected Kähler manifold with a Killing vector field* X *such that* zero(X) *is a discrete set, then* $b_{2k}(M^n) = (-1)^k \chi^k(M^n)$.

PROOF. Since λ_α is even, 2.1 implies $b_{2k+1}(M^n) = 0$ and $b_{2k}(M^n)$ is equal to the number of points of zero(X) such that $\lambda_\alpha = 2k$, $k = 1, \cdots, n$ [5]. This is just the number of points such that $s(p_\alpha, i) = k$. Thus, by 3.4 we have

$$b_{2k}(M^n) = (-1)^k \chi^k(M^n),\ k = 1, \cdots, n. \quad \text{Q.E.D.}$$

5.2 THEOREM [8]. *If* M^n *and* X *are as in* 5.1, *then* $h^{p,q} = 0$, $p \neq q$.

PROOF. Let $h^{p,q} = \dim_c H^{p,q}(M^n)$. Recall (2.2) that $b_{2k+1}(M^n) = 0$, $k = 0, \cdots, n$. Thus, $h^{p,q} = 0$, $p + q$ odd.

Now from 5.1 $\chi^0(M^n) = b_0(M^n) = h^{0,0}$. Thus by 3.2

$$h^{0,0} - h^{0,1} + h^{0,2} + \cdots + (-1)^n h^{0,n} = h^{0,0} \quad \text{and} \quad h^{0,q} = 0 \quad \text{for } q \text{ odd.}$$

Thus, we have $\Sigma_{\text{even}} h^{0,j}$ and since $h^{0,j} \geqslant 0$, we obtain $h^{0,j} = 0$, $j = 1, \cdots, n$.

Assume $h^{p,q} = 0$, $p \neq q, p \leqslant l$. We show $h^{l+1,q} = 0$, $q \neq l+1$. By 5.1

$$h^{l+1,0} - h^{l+1,1} + \cdots + (-1)^{l+1} h^{l+1,l+1} + \cdots + (-1)^n h^{l-1,n} = (-1)^{l+1} b_{2(l+1)}(M^n).$$

But [7] $b_{2(l+1)}(M^n) = \Sigma_{p+q=2(l+1)} h^{p,q}$. Thus, by the induction assumption $b_{2(l+1)}(M^n) = h^{l+1,l+1}$. Therefore

$$h^{l+1,0} - h^{l+1,1} + \cdots + (-1)^l h^{l+1,l} + (-1)^{l+2} h^{l+1,l+2} + \cdots + (-1)^n h^{l+1,n} = 0,$$

but $h^{l+1,j} = 0$, $l + j + 1 =$ odd. Therefore, $h^{l+1,q} = 0$, $l + 1 \neq q$. Q.E.D.

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